Slip Effects on Steady Flow Through a Stenosed Blood Artery

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Abstract – The effect of slip boundary conditions on the blood flow taking blood as a Casson fluid has been studied. It is observed that axial velocity, volumetric flow rate and pressure gradient decrease along the radial distance as the slip length increases but the wall shear stress increases with increase in slip length. All the results derived are presented analytically and graphically by selecting suitable parameters.

Key Words – Non – Newtonian Fluid, Blood Flow, Casson Fluid, Slip Boundary Condition, Shear Stress, Stenosed Artery, Yield Stress.

1 INTRODUCTION

It is now a well proved fact that stenosis has become a serious threat to the life which needs an immediate attention. The artery becomes stenosed when its wall becomes fatty due to abnormal development along the lumen of the wall. Because of this stenosis the hemodynamic behaviour of the blood flow is badly affected. The stenosis of the artery gives rise to many medical problems like stroke, heart attack and serious circulatory disorders.

Many researchers have proved that the blood shows a very interesting behaviour. It behaves like a Newtonian fluid at high shear rate and it behaves like a non – Newtonian fluid at low shear rate. Y. Nuber (1971) studied the blood flow, slip and viscometry and the study showed that the viscosity indications would exhibit a flow dependent behaviour of much the same pattern as the actual indications supplied by the usual viscometers if the slip function is of plausible form. M.D. Despande et al. (1976) discussed the steady laminar flow through modelled vascular stenoses and compared the theoretical results with available experimental values. J.B. Shukla et. al. (1980) analyzed the effects of stenosis on non – Newtonian flow of the blood in an artery and showed that the increments in the size of the stenosis produce small increments in the flow resistance and wall shear stress as the blood shows a non – Newtonian behaviour. K. Haldar (1985) studied the effects of stenosis shape on blood flow resistance and proved that the variations in the stenosis shape may decrease the flow resistance but the symmetric stenosis gives maximum resistance to flow. L.M. Srivastava (1985) also discussed the flow of couple stress fluid through stenotic blood vessels and showed that the flow resistance and wall shear stress in case of mild stenosis of non – Newtonian blood are increased over those with no stenosis by 60% and 62% respectively in comparison to the Newtonian fluid. J.C. Misra et al. (1993) presented a non – Newtonian model for blood flow through arteries under stenotic conditions and gave a qualitative analysis for the frequency variations of flow rate at various points of the artery, phase velocities and transmission per wavelength. J.C. Misra et al. (2007) discussed the role of slip velocity in blood flow through stenosed arteries considering the blood as a Herschel – Bulkley fluid and investigated the influence of the slip at the wall of the vessel with mild, moderate and severe stenoses. D. Biswas et al. (2011) gave a non – Newtonian model to study the steady blood flow through a stenosed artery taking blood as a Herschel – Bulkley fluid and observed that axial velocity, flow rate increase with slip and decrease with yield stress.

2 MATHEMATICAL FORMULATION

Laminar steady flow of an incompressible Casson fluid through a cylindrical artery having axially symmetric stenosis is considered. The geometry of the artery is described below:

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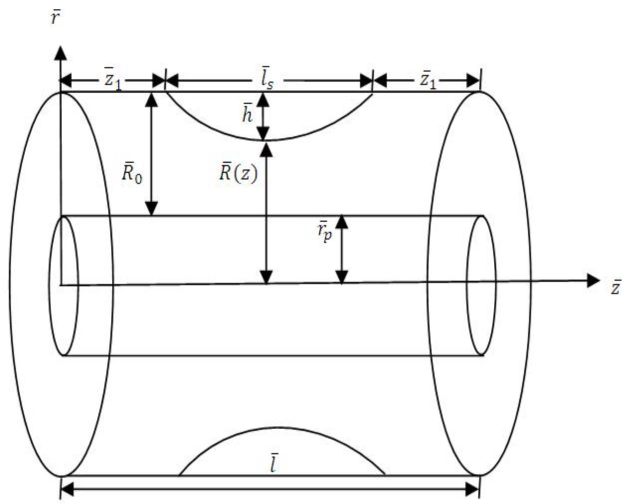
ISSN 2229-5518



Let $\bar{R}(\bar{z})$ be the radius of the artery in the stenotic region and $\bar{R}_0$ in the non – stenotic area given as (Young, 1968):

$$\bar{R}(\bar{z}) = \begin{cases} \bar{R}_0 - \frac{\bar{h}}{2}\left[1 + \cos\frac{2\pi}{\bar{l}_s}\left(\bar{z}_1 + \bar{l}_s - \bar{z}\right)\right]; & \bar{z}_1 \le \bar{z} \le \bar{z}_1 + \bar{l}_s \\ \bar{R}_0 & ; \text{ otherwise} \end{cases} \tag{2.1}$$

where $\bar{h}, \bar{l}_s$ and $\bar{z}_1$ are the maximum height, length and the location of the stenosis in the artery with whole length $\bar{l}$. Also, let $\bar{r}$ and $\bar{z}$ are the radial and axial coordinates.

With above considerations, the equations of motion for the blood can be given as

$$-\frac{\partial \bar{p}}{\partial \bar{z}} + \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}\bar{\tau}_c) = 0 \tag{2.2}$$

$$\frac{\partial \bar{p}}{\partial \bar{r}} = 0 \tag{2.3}$$

Here $\bar{p}$ denotes the pressure at any point and $\bar{\tau}_c$ gives the shear stress of Casson fluid with the following simplified constitutive equations:

$$F(\bar{\tau}_c) = -\frac{\partial \bar{v}_c}{\partial \bar{r}} = \frac{1}{\bar{k}_c}\left(\bar{\tau}_c^{1/2} - \bar{\tau}_0^{1/2}\right)^2 \quad \text{for } \bar{\tau}_c \ge \bar{\tau}_0 \tag{2.4}$$

$$\frac{\partial \bar{v}_c}{\partial \bar{r}} = 0 \qquad \text{for } \bar{\tau}_c \le \bar{\tau}_0 \tag{2.5}$$

where $\bar{v}_c$ is the axial velocity of fluid, $\bar{\tau}_0$ represents the yield stress and $\bar{k}_c$ is the fluid viscosity.

The flow is subject to slip boundary conditions as follows:

$$\left.\begin{aligned} \bar{v}_c &= \bar{\beta}\frac{\partial \bar{v}_c}{\partial \bar{r}} && \text{at } \bar{r} = \bar{R}(\bar{z}) \\ \bar{\tau}_c &= \text{Finite value} && \text{at } \bar{r} = 0 \end{aligned}\right\} \tag{2.6}$$

where $\bar{\beta}$ represents the slip length in the axial direction

Using following non – dimensional quantities:

$$R(z) = \frac{\bar{R}(\bar{z})}{\bar{R}_0},\ z = \frac{\bar{z}_1 + \bar{l}_s - \bar{z}}{\bar{l}_s},\ r = \frac{\bar{r}}{\bar{R}_0},\ H = \frac{\bar{h}}{\bar{R}_0},\ \frac{\partial p}{\partial z} = \frac{\partial \bar{p}/\partial \bar{z}}{\bar{p}_0},\ \tau_c = \frac{\bar{\tau}_c}{\bar{p}_0 \bar{R}_0/2},\ \tau_0 = \frac{\bar{\tau}_0}{\bar{p}_0 \bar{R}_0/2},\ v_c = \frac{\bar{v}_c}{\bar{p}_0 \bar{R}_0^2/2\bar{k}_c},\ \beta = \frac{\bar{\beta}}{\bar{R}_0}. \tag{2.7}$$

where $\bar{p}_0$ is the absolute typical pressure gradient.

The non – dimensional radius of the stenotic area of the artery is

$$R(z) = \begin{cases} 1 - H\cos^2 \pi z; & 0 \le z \le 1 \\ 1 & ; \text{ otherwise} \end{cases} \tag{2.8}$$

The non – dimensional forms of the equations of the motion (2.2) and (2.3) are

$$-2\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_c) = 0 \tag{2.9}$$

$$\frac{\partial p}{\partial r} = 0 \tag{2.10}$$

The constitutive equations (2.4) and (2.5) of the Casson fluid in the dimensionless forms, can be written as

$$-\frac{\partial v_c}{\partial r} = \left(\tau_c^{1/2} - \tau_0^{1/2}\right)^2 \qquad \text{for } \tau_c \ge \tau_0 \tag{2.11}$$

$$\frac{\partial v_c}{\partial r} = 0 \qquad \text{for } \tau_c \le \tau_0 \tag{2.12}$$

The non – dimensional boundary conditions are

$$\left.\begin{aligned} v_c &= \beta\frac{\partial v_c}{\partial r} && \text{at } r = R(z) \\ \tau_c &= \text{Finite value} && \text{at } r = 0 \end{aligned}\right\} \tag{2.13}$$

Using boundary conditions (2.13) in equation (2.9), we get the expressions for the shear stress τ_c and wall shear stress τ_R in the following forms:

$$\tau_c = -r\frac{\partial p}{\partial z} \tag{2.14}$$

$$\tau_R = -R(z)\frac{\partial p}{\partial z} \tag{2.15}$$

From equations (2.14) and (2.15),

$$\frac{\tau_c}{\tau_R} = \frac{r}{R} \tag{2.16}$$

where $R = R(z)$

3 METHOD OF SOLUTION

Integrating equation (2.11) using equations (2.13) to (2.15), the velocity profile for $r_p \le r \le R(z)$ where $r_p = \frac{\bar{r}_p}{\bar{R}_0}$ is the non – dimensional radius of the plug flow region, is

$$v_c = \frac{R}{2\tau_R}\left[(\tau_R^2 - \tau_c^2) - \frac{8}{3}\tau_0^{1/2}\left(\tau_R^{3/2} - \tau_c^{3/2}\right) + 2\tau_0(\tau_R - \tau_c)\right] - \beta\left(\tau_R^{1/2} - \tau_0^{1/2}\right)^2 \quad (3.1)$$

Within pug flow region i.e. $0 \le r \le r_p$, $\tau_c = \tau_0$ at $r = r_p$.

Then from equation (3.1), the plug flow velocity is

$$v_p = \frac{R}{2\tau_R}\left(\tau_R^2 - \frac{1}{3}\tau_0^2 - \frac{8}{3}\tau_0^{1/2}\tau_R^{3/2} + 2\tau_0\tau_R\right) - \beta\left(\tau_R^{1/2} - \tau_0^{1/2}\right)^2 \quad (3.2)$$

The non – dimensional volumetric flow rate in the form for the region $0 \le r \le R(z)$ is calculated as

$$Q = 4\int_0^R r v(r) dr$$

$$= 4\int_0^{r_p} r v_p dr + 4\int_{r_p}^{R} r v_c dr$$

Hence

$$Q = \frac{2R^3}{\tau_R^3}\left(\frac{1}{4}\tau_R^4 - \frac{4}{7}\tau_0^{1/2}\tau_R^{7/2} + \frac{1}{3}\tau_0\tau_R^3 - \frac{1}{84}\tau_0^4\right) - 2R^2\beta\left(\tau_R^{1/2} - \tau_0^{1/2}\right)^2 \quad (3.3)$$

If $\tau_0 \ll \tau_R$ i.e. $\frac{\tau_0}{\tau_R} \ll 1$, then equation (3.3) becomes

$$Q = \frac{R^3}{2}\left(\tau_R - \frac{16}{7}\tau_0^{1/2}\tau_R^{1/2} + \frac{4}{3}\tau_0\right) - 2R^2\beta\left(\tau_R^{1/2} - \tau_0^{1/2}\right)^2 \quad (3.4)$$

which can also be used to get the wall shear stress for the stenosed artery given as

$$\tau_R = \left[\frac{4}{7}\left(\frac{2R-7\beta}{R-4\beta}\right)\tau_0^{1/2} + \left\{\frac{2Q}{R^2(R-4\beta)} + \frac{16}{49}\frac{(2R-7\beta)^2}{(R-4\beta)^2}\tau_0 - \frac{4}{3}\left(\frac{R-3\beta}{R-4\beta}\right)\tau_0\right\}^{1/2}\right]^2 \quad (3.5)$$

For an artery without stenosis i.e. $R(z) = R_0$, the wall shear stress is given as

$$\tau_R = \left[\frac{4}{7}\left(\frac{2R_0-7\beta}{R_0-4\beta}\right)\tau_0^{1/2} + \left\{\frac{2Q}{R_0^2(R_0-4\beta)} + \frac{16}{49}\frac{(2R_0-7\beta)^2}{(R_0-4\beta)^2}\tau_0 - \frac{4}{3}\left(\frac{R_0-3\beta}{R_0-4\beta}\right)\tau_0\right\}^{1/2}\right]^2 \quad (3.6)$$

Now using equation (3.5) in equation (2.15), we can compute the pressure gradient as

$$\frac{\partial p}{\partial z} = -\frac{1}{R}\left[\frac{4}{7}\left(\frac{2R-7\beta}{R-4\beta}\right)\tau_0^{1/2} + \left\{\frac{2Q}{R^2(R-4\beta)} + \frac{16}{49}\frac{(2R-7\beta)^2}{(R-4\beta)^2}\tau_0 - \frac{4}{3}\left(\frac{R-3\beta}{R-4\beta}\right)\tau_0\right\}^{1/2}\right]^2 \quad (3.7)$$

4 RESULTS AND DISCUSSION

The velocity profile for the axial velocity in the non – plug flow region has been obtained in equation (3.1) and results are analyzed using graphs in figures 1(a) and 1(b).

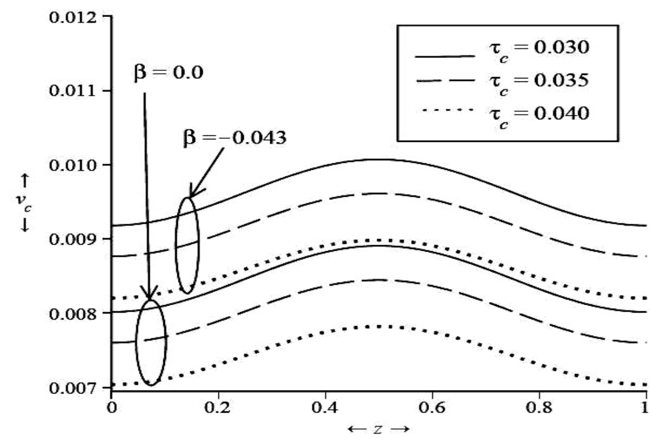


Figure 1 (a): Variation of Axial Velocity Along Axial Distance for Different Values of the Shear Stress τ_c and Slip Length β with Some Fixed Values $\tau_R = 0.070, \tau_0 = 0.010, H = 0.1$

Figures 1(a) shows the variations of the axial velocity along the axial distance z for the different values of the shear stress τ_c and slip length β with some fixed values $\tau_R = 0.070$, $\tau_o = 0.010$ and $H = 0.1$. It is clear that the axial velocity first increases and then decreases after attaining a maximum value along the axial distance z. It also clarifies that the axial velocity increases whenever the velocity slip β increase and it decreases for the increasing values of the shear stress.

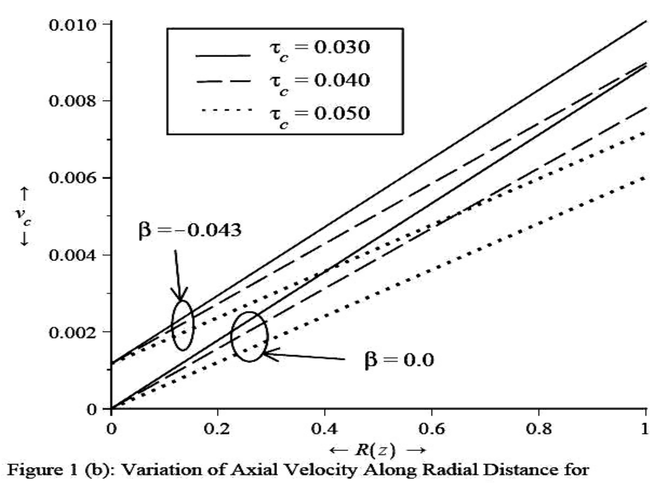


Figure 1 (b): Variation of Axial Velocity Along Radial Distance for Different Values of the Shear Stress τ_c and Slip Length β with Some Fixed Values $\tau_R = 0.070, \tau_0 = 0.010$.

ISSN 2229-5518

Figure 1(b) shows the variations of the axial velocity along the radial distance R(z) for the different values of the shear stress τ_c and slip length β with some fixed values $\tau_R = 0.070$ and $\tau_o = 0.010$. Graph shows that the axial velocity is increasing along the radial distance. Also that the axial velocity increases when the velocity slip increases and it decreases as the shear stress increases.

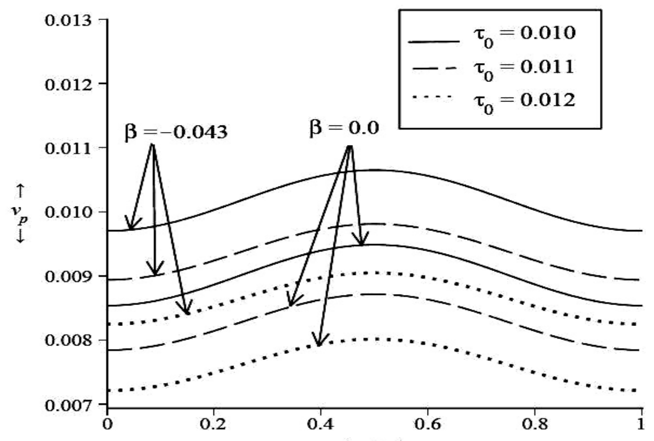


Figure 2 (a): Variation of Plug Flow Velocity Along Axial Distance for Different Values of the Yield Stress τ_0 and Slip Length β with Some Fixed Values $\tau_R = 0.070, H = 0.1$.

The axial velocity for the plug flow region obtained through equation (3.2) has been analyzed in figure 2(a) which shows the variations of the plug flow velocity along the axial distance z taken for the different values of the yield stress τ_0 and slip length β with fixed values τ_R = 0.070 and H = 0.1. It is observed here that the plug flow velocity is showing wavy variations along the axial distance z. Also the plug flow velocity increases as the velocity slip increases and it decreases when the yield stress increases.

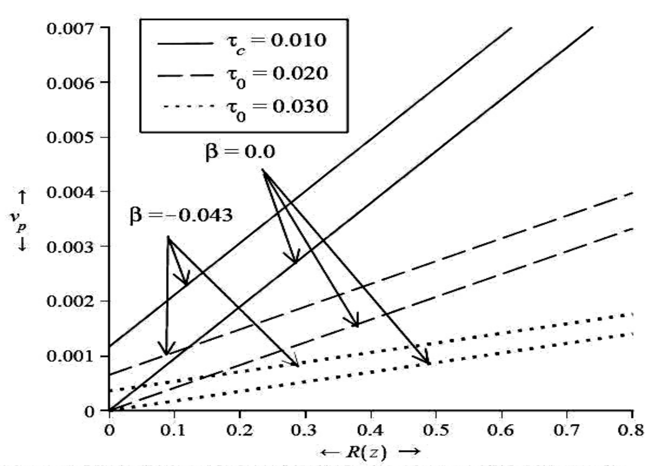


Figure 2 (b): Variation of Plug Flow Velocity Along Radial Distance for Different Values of the Yield Stress τ_0 and Slip Length β with Some Fixed Fixed Values $\tau_R = 0.070$.

Figure 2(b) shows the variations of the axial velocity along radial distance R(z) for the different values of the yield stress τ_0 and slip length β with other fixed values $\tau_R = 0.070$. It shows that the plug flow velocity increases along the radial distance and it decreases when the yield stress increases. Also the plug flow velocity increases as the velocity slip increases. It is to be noted here that for the greater values of the yield stress, the plug flow velocity increases slowly as the velocity slip increases as compared to the lower values of the yield stress.

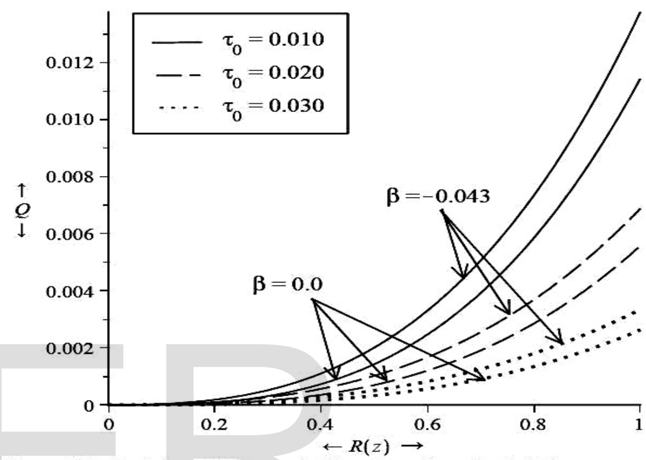


Figure 3(a): Variation of Volumetric Flow Rate Along Radial Distance for Different Values of the Yeild Stress τ_0 and Slip Legth β with Some Fixed Value $\tau_R = 0.070$.

The volumetric flow rate derived through equation (3.4) has been graphically presented in figures 3(a) and 3(b). Figure 3(a) shows the variations of the volumetric flow rate along the radial distance R(z) for the various values of the yield stress τ_0 and slip length β with a fixed value τ_R = 0.070. Clearly the volumetric flow rate increases along the radial distance. It is observed that the volumetric flow rate increases as the velocity slip increases but it decreases when the yield stress increases. Also we see that the volumetric flow rate increases at a little slower rate for the greater values of the yield stress in comparison to the lower yield stress.

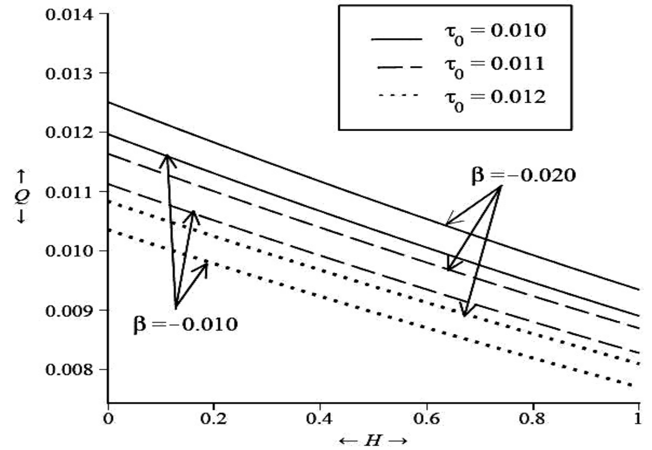


Figure 3(b): Variation of Volumetric Flow Rate Along the Stenosis Height for Different Values of the Yeild Stress τ_0 and Slip Length β with Some Fixed Value $\tau_R = 0.070, z = 0.4$.

Figure 3(b) presents the variations of the volumetric flow rate are shown along the height of the stenosis H for the different values of the yield stress τ_0 and wall slip length β with fixed values $\tau_R = 0.070$ and $z = 0.4$. It is obvious that the volumetric flow rate slows down along the height of the stenosis with increments in yield stress but it increases when the velocity slip increases.

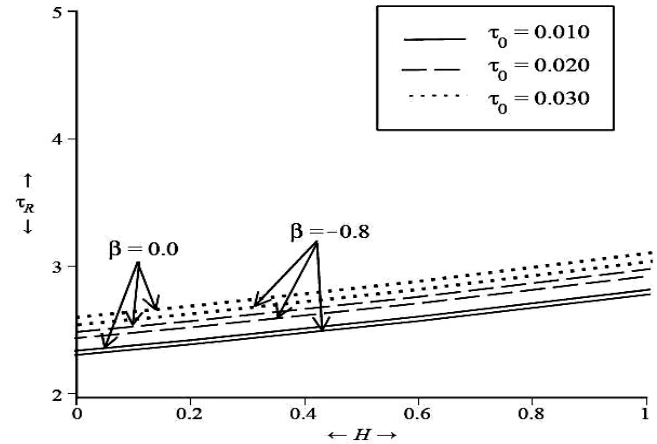


Figure 4(b): Variation of Wall Shear Stress Along the Stenosis Height for Different Values of the Yeild Stress τ_0 and Slip Length β with Some Fixed Values $Q = 1, z = 0.4$.

Figure 4(b) shows the variations of the of the wall shear stress along the height of the stenosis H for the different values of the yield stress τ_0 and wall slip length β with some fixed values $Q = 1$ and $z = 0.4$. It is clear that the wall shear stress increases along the height of the stenosis. Also the wall shear stress increases as the yield stress increases and it decreases when the velocity slip increases.

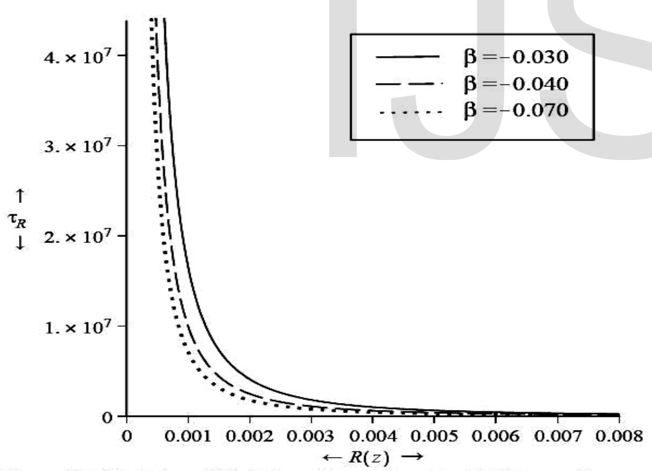


Figure 4(a): Variation of Wall Shear Stress Along Radial Distance for Different Values of the Slip Legth β with Some Fixed Values $Q = 1$ and Yield Stress τ_0.

Figure 4(a) explains the variations of the wall shear stress obtained in equation (3.5) along the radial distance R(z) for the different values of the slip length β with a fixed value $Q = 1$. It shows that the wall shear stress decreases along the radial distance. Also the wall shear stress decreases when velocity slip increases.

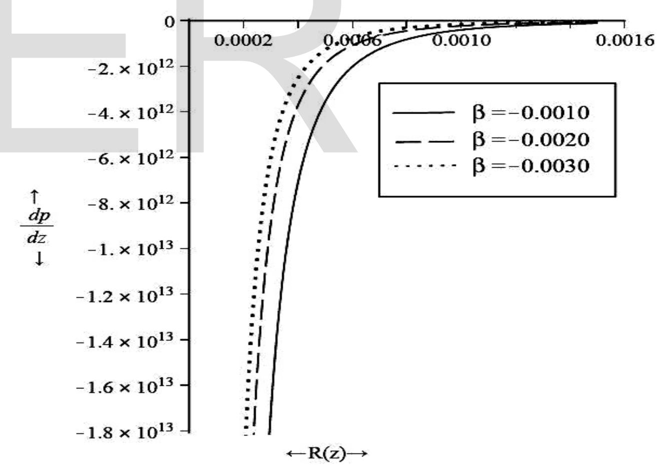


Figure 5(a): Variation of Pressure Gradient Along Radial Distance $R(z)$ for Differentt Values of Slip Length β with Some fixed Value $Q = 1$.

The variations of the pressure gradient obtained in equation (3.7) are shown in figures 5(a) and 5(b). Figure 5(a) shows that the variations of the pressure gradient along the radial distance R(z) for the different values of the slip length β with a fixed value $Q = 1$. Obviously the pressure gradient increases along the radial distance. Also it increases as the velocity slip increases.

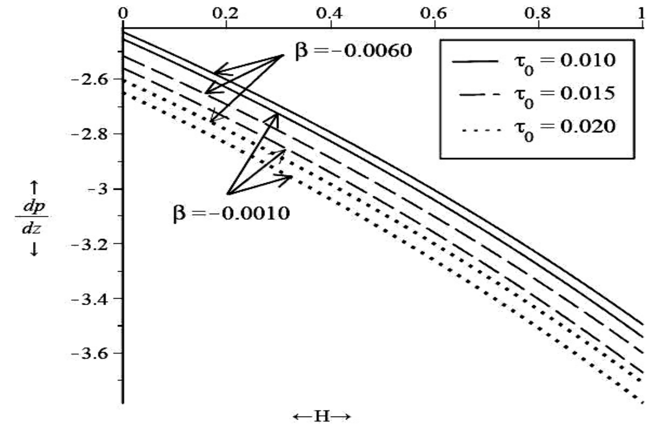


Figure 5(b): Variation of Pressure Gradient Along the Stenosis Height for Different Values of the Yeild Sress τ_0 and Slip Length β with Some Fixed Value $Q = 1, z = 0.4$..

Figure 5(b) gives the variations of the pressure gradient along the height of the stenosis H for the various values of the yield stress τ_0 and wall slip length β with some fixed values Q = 1 and z = 0.4. It is observed that the pressure gradient decreases greatly along the height of the stenosis. Also the pressure gradient decreases whenever the yield stress increases and it increases as the velocity slip increases.

5 CONCLUSION

In the present study authors made an attempt to present the theoretical observations of the different flow features by considering a stenosed artery with blood behaving like a Casson fluid. The results are explained analytically and graphically by choosing some suitable parameters. The graphical analysis of the study reveals that the axial velocity is showing the wave like variations along the axial distance z and for increments in velocity slip it increases in both plug flow and non – plug flow domains. Also the axial velocity increases along the radial distance as the slip length increases in both plug flow and non – plug flow regions. The volumetric flow rate increases along the radial distance as the velocity slip increases. The axial velocity and the volumetric flow rate decrease when the yield stress increase. It is observed that the plug flow velocity and the volumetric flow rate increase gradually for the greater values of the yield stress as compared to the lower yield stress. The wall shear stress decreases and the pressure gradient increases along the radial distance as the velocity slip increases. The analysis regarding the effect of the stenosis over other flow properties like volumetric flow rate, wall shear stress and pressure gradient has also been done. The volumetric flow rate and the pressure gradient decrease when the yield stress increases but they increase with increments in velocity slip along the height of the stenosis. Also the wall shear stress increases as the yield stress increases and it decreases when the velocity slip increases along the height of the stenosis.

6 REFERENCES

[1] Young, D.F. (1968): Effect of time dependent stenosis on flow through a tube, J. Engg. for Ind., Trans of ASME, 90, 248-254.

[2] Nubar, Y. (1971): Blood flow, slip, and viscometry, Biophysical Journal, vol. 11, no. 3, pp. 252 – 264.

[3] Despande, M.D.; Giddens, D.P. and Mabon, R.F. (1976): Steady laminar flow through modelled vascular stenoses, Journal of Biomechanics, Vol.9, pp. 165 – 174.

[4] Shukla, J.B.; Parihar, R.S. and Rao, B.R.P. (1980): Effects of stenosis on non – Newtonian flow of the blood in an artery, Bull. Math. Bio., Vol. 42, pp. 283 – 294.

[5] Haldar, K. (1985): Effects of the shape of stenosis on the resistance to blood flow through an artery, Bulletin of Mathematical Biology, Vol. 47, No. 4, pp. 545 – 550.

[6] Srivastava, L.M. (1985): Flow of couple stress fluid through stenotic blood vessels, Journal of Biomechanics, vol. 18, no. 7, pp. 479 – 485.

[7] Misra, J.C.; Patra, M.K. and Misra, S.C. (1993): A non-Newtonian fluid model for blood flow through arteries under stenotic conditions, Journal of Biomechanics, vol. 26, no. 9, pp. 1129 – 1141.

[8] Misra, J.C. and Shit, G.C. (2007): Role of slip velocity in blood flow through stenosed arteries: a non-Newtonian model, Journal of Mechanics in Medicine and Biology, vol. 7, no. 3, pp. 337 – 353.

[9] Biswas, D. and Laskar, R.B. (2011): Steady flow of blood through stenosed artery: A non – Newtonian fluid model, Assam Univ. J. Sc. & Tech., Vol. 7, No. 11, pp. 144 – 153.